§2.1 - LINEAR FIRST-ORDER
(METHOD OF INTEGRATING FACTORS)

In general, linear first-order equations can be written

\[ P(t) \frac{dy}{dt} + Q(t)y = G(t) \]

Although in order to apply some methods of solution we may have to solve and rearrange them into "standard form":

\[ \frac{dy}{dt} + p(t)y = q(t) \]

\[ \text{[E]} \text{ Solve } (4 + t^2) \frac{dy}{dt} + 2ty = 4t. \]

Solution: Note the left hand side is just

\[ \frac{d}{dt}[(4 + t^2)y] = (4 + t^2) \frac{dy}{dt} + 2ty \]

The right hand side is just a function of \( t \), so we can integrate both sides in \( t \):

\[ \int \frac{d}{dt}[(4 + t^2)y] \, dt = \int 4t \, dt \]

\[ \Rightarrow (4 + t^2)y = 2t^2 + C \]

\[ \Rightarrow y = \frac{2t^2 + C}{4 + t^2} \]

//Note this is never 0

\[ \text{[!]} \text{ Note how important it was for us to add } C \text{ when we did!} \]

If we waited until the end, solved for \( y \), and then added \( +C \), we would get \( y = \frac{2t^2 + C}{4 + t^2} \), which is not the same!

\[ \text{[E]} \text{ Solve } \frac{dy}{dt} + \frac{2t}{4t^2} \, y = \frac{4t}{4 + t^2} \]

Solution: The LHS doesn't seem to be the derivative of anything we are familiar with. Obviously the answer should be \( y = \frac{2t^2 + C}{4 + t^2} \) as before, but how do we get there?

\[ \text{[!]} \text{ Integrating Factor!} \]
INTEGRATING FACTOR

Note that if we have the ODE
\[
\frac{dy}{dt} + \frac{2t}{4+t^2} y = \frac{4t}{4+t^2}
\]
we can multiply both sides by \(4+t^2\) to get
\[
(4+t^2) \frac{dy}{dt} + 2ty = 4t,
\]
which we can solve. In this example, \(4+t^2\) was an \underline{integrating factor} - an expression we can multiply both sides of the ODE by in order to integrate.

\[\frac{dy}{dt} + \frac{1}{2} y = \frac{1}{2} e^{t/3}\]

\[\boxed{\text{Solution: This doesn't look familiar, but maybe it has an integrating factor.}}\]

\[H(t) \frac{dy}{dt} + \frac{1}{2} H(t) y = \frac{1}{2} H(t) e^{t/3}\]

We would need:
\[
\frac{d}{dt} \left( H(t) y \right) = H(t) \frac{dy}{dt} + \frac{1}{2} H(t) y
\]

\[\Rightarrow H'(t) = \frac{1}{2} H(t) \Rightarrow H(t) = Ce^{t/2}\]

\[\text{//Note: there are lots of integrating factors (just multiply by a constant) but we only need 1. Take } C=1.\]

Taking \(H(t) = e^{t/2}\), we can now integrate:
\[
\frac{dy}{dt} + \frac{1}{2} y = \frac{1}{2} e^{t/3} \iff e^{t/2} \frac{dy}{dt} + \frac{1}{2} e^{t/2} y = \frac{1}{2} e^{t/2} e^{t/3}
\]

\[\Rightarrow \frac{d}{dt} \left( e^{t/2} y \right) = \frac{1}{2} e^{5t/2}\]

\[\Rightarrow \int \frac{d}{dt} \left( e^{t/2} y \right) dt = \int \frac{1}{2} e^{5t/2} dt \Rightarrow e^{t/2} y = \frac{3}{5} e^{5t} + C\]
In general, if we have
\[
\frac{dy}{dt} + p(t) y = g(t)
\]
\[
\Rightarrow \quad \mu(t) \frac{dy}{dt} + \mu(t) p(t) y = \mu(t) g(t)
\]
We want \[
\frac{d}{dt} (\mu(t) y) \Rightarrow \mu'(t) = \mu(t) p(t)
\]
This is another ODE, but we can solve it!
\[
\frac{\mu'(t)}{\mu(t)} = p(t) \Rightarrow \int \frac{\mu'(t)}{\mu(t)} dt = \int p(t) dt
\]
\[
\Rightarrow \log |\mu(t)| = \int p(t) dt + C
\]
\[
\Rightarrow \mu(t) = C e^{\int p(t) dt}
\]
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\[
\Rightarrow \mu(t) = C e^{\int p(t) dt}
\]

Assuming we can do the integration, we have
\[
\frac{dy}{dt} + p(t) y = g(t) \Rightarrow \frac{d}{dt} (\mu(t) y) = g(t) \mu(t)
\]
\[
= \int g(t) \mu(t) dt = \int g(t) \mu(t) dt
\]
\[
\Rightarrow \mu(t) y = \int g(t) \mu(t) dt + C
\]
\[
\Rightarrow y = \frac{1}{\mu(t)} \left( \int g(t) \mu(t) dt + C \right)
\]
\[
= e^{-\int p(t) dt} \left( \int g(t) e^{\int p(t) dt} dt + C \right)
\]
We can use this like a black box:

\[ \frac{dy}{dt} + p(t) y = g(t), \text{ then} \]

\[ y = e^{-\int p(t) \, dt} \left( \int g(t) e^{\int p(t) \, dt} \, dt + C \right) \]

\[ ty' + 2y = 4t^2, \quad y(1) = 2. \]

Solution: Rewrite in "standard form":

\[ y' + \frac{2}{t} y = 4t \]

\[ \Rightarrow y = e^{-\int \frac{2}{t} \, dt} \left( \int 4t e^{\int \frac{2}{t} \, dt} \, dt + C \right) \]

\[ = e^{-2\log t} \left( \int 4t e^{2\log t} \, dt + C \right) \]

\[ = t^{-2} \left( \int 4t^2 \, dt + C \right) \]

\[ = t^{-2} \left( t^4 + C \right) = t^2 + \frac{C}{t^2} \]

\[ y(1) = 2 \Rightarrow 1^2 + \frac{C}{1^2} = 2 \Rightarrow C = 1, \]

So \[ y(t) = t^2 + t^{-2} \]